Chapter 6

**Matched Filter and the Radar Ambiguity Function**

6.1. The Matched Filter SNR

The most unique characteristic of the matched filter is that it produces the maximum achievable instantaneous SNR at its output when a signal plus additive white noise are present at the input. The noise does not need to be Gaussian. The peak instantaneous SNR at the receiver output can be achieved by matching the radar receiver transfer function to the received signal. We will show that the peak instantaneous signal power divided by the average noise power at the output of a matched filter is equal to twice the input signal energy divided by the input noise power, regardless of the waveform used by the radar. This is the reason why matched filters are often referred to as optimum filters in the SNR sense. Note that the peak power used in the derivation of the radar equation (SNR) represents the average signal power over the duration of the pulse, not the peak instantaneous signal power as in the case of a matched filter. In practice, it is sometimes difficult to achieve perfect matched filtering. In such cases, sub-optimum filters may be used. Due to this mismatch, degradation in the output SNR occurs.

Consider a radar system that uses a finite duration energy signal \( s_i(t) \). Denote the pulse width as \( \tau \), and assume that a matched filter receiver is utilized. The main question that we need to answer is: What is the impulse, or frequency, response of the filter that maximizes the instantaneous SNR at the output of the receiver when a delayed version of the signal \( s_i(t) \) plus additive white noise is at the input?

The matched filter input signal can then be represented by

\[
x(t) = C \cdot s_i(t - t_1) + n_i(t)
\]  

(6.1)
where \( C \) is a constant, \( t_1 \) is an unknown time delay proportional to the target range, and \( n_r(t) \) is input white noise. Since the input noise is white, its corresponding autocorrelation and Power Spectral Density (PSD) functions are given, respectively, by

\[
\tilde{R}_{n_r}(t) = \frac{N_0}{2} \delta(t) \tag{6.2}
\]

\[
\tilde{S}_{n_r} (\omega) = \frac{N_0}{2} \tag{6.3}
\]

where \( N_0 \) is a constant. Denote \( s_o(t) \) and \( n_o(t) \) as the signal and noise filter outputs. More precisely, we can define

\[
y(t) = C s_o(t - t_1) + n_o(t) \tag{6.4}
\]

where

\[
s_o(t) = s_j(t) \ast h(t) \tag{6.5}
\]

\[
n_o(t) = n_j(t) \ast h(t) \tag{6.6}
\]

The operator \(( \ast )\) indicates convolution, and \( h(t) \) is the filter impulse response (the filter is assumed to be linear time invariant).

Let \( R_{h}(t) \) denote the filter autocorrelation function. It follows that the output noise autocorrelation and PSD functions are

\[
\tilde{R}_{n_o}(t) = \tilde{R}_{n_r}(t) \ast R_{h}(t) = \frac{N_0}{2} \delta(t) \ast R_{h}(t) = \frac{N_0}{2} R_{h}(t) \tag{6.7}
\]

\[
\tilde{S}_{n_o} (\omega) = \tilde{S}_{n_r}(\omega) |H(\omega)|^2 = \frac{N_0}{2} |H(\omega)|^2 \tag{6.8}
\]

where \( H(\omega) \) is the Fourier transform for the filter impulse response, \( h(t) \). The total average output noise power is equal to \( \tilde{R}_{n_o}(t) \) evaluated at \( t = 0 \). More precisely,

\[
\tilde{R}_{n_o}(0) = \frac{N_0}{2} \int_{-\infty}^{\infty} |h(u)|^2 du \tag{6.9}
\]

The output signal power evaluated at time \( t \) is \( |C s_o(t - t_1)|^2 \), and by using Eq. (6.5) we get
A general expression for the output SNR at time \( t \) can be written as

\[
s_n(t-t_1) = \int_{\infty}^{-\infty} s_i(t-t_1-u) \cdot h(u) \, du \quad (6.10)
\]

Substituting Eqs. (6.9) and (6.10) into Eq. (6.11) yields

\[
SNR(t) = \frac{\left| Cs_n(t-t_1) \right|^2}{R_n(0)} \quad (6.11)
\]

where the Schwartz inequality states that

\[
\int_{\infty}^{-\infty} P(x)Q(x)dx \leq \int_{\infty}^{-\infty} |P(x)|^2 \, dx \cdot \int_{\infty}^{-\infty} |Q(x)|^2 \, dx \quad (6.13)
\]

where the equality applies only when \( P = kQ^* \), where \( k \) is a constant and can be assumed to be unity. Then by applying Eq. (6.13) on the numerator of Eq. (6.12), we get

\[
SNR(t) \leq \frac{C^2 \int_{\infty}^{-\infty} |s_i(t-t_1-u)|^2 \, du \cdot \int_{\infty}^{-\infty} |h(u)|^2 \, du}{N_0/2 \int_{\infty}^{-\infty} |h(u)|^2 \, du} \quad (6.14)
\]

\[
= \frac{2C^2 \int_{\infty}^{-\infty} |s_i(t-t_1-u)|^2 \, du}{N_0}
\]
Eq. (6.14) tells us that the peak instantaneous SNR occurs when equality is achieved (i.e., from Eq. (6.13) \( h = k s_i^* \)). More precisely, if we assume that equality occurs at \( t = t_0 \), and that \( k = 1 \), then

\[
h(u) = s_i^*(t_0 - t_1 - u)
\]

(6.15)

and the maximum instantaneous SNR is

\[
SNR(t_0) = \frac{2C^2 \int |s_i(t_0 - t_1 - u)|^2 \, du}{N_0}
\]

(6.16)

Eq. (6.16) can be simplified using Parseval’s theorem,

\[
E = C^2 \int |s_i(t_0 - t_1 - u)|^2 \, du
\]

(6.17)

where \( E \) denotes the energy of the input signal; consequently we can write the output peak instantaneous SNR as

\[
SNR(t_0) = \frac{2E}{N_0}
\]

(6.18)

Thus, we can draw the conclusion that the peak instantaneous SNR depends only on the signal energy and input noise power, and is independent of the waveform utilized by the radar.

Finally, we can define the impulse response for the matched filter from Eq. (6.15). If we desire the peak to occur at \( t_0 = t_1 \), we get the non-causal matched filter impulse response,

\[
h_{nc}(t) = s_i^*(-t)
\]

(6.19)

Alternatively, the causal impulse response is

\[
h_c(t) = s_i^*(\tau - t)
\]

(6.20)

where in this case, the peak occurs at \( t_0 = t_1 + \tau \). It follows that the Fourier transforms of \( h_{nc}(t) \) and \( h_c(t) \) are given, respectively, by

\[
H_{nc}(\omega) = S_i^*(\omega)
\]

(6.21)

\[
H_c(\omega) = S_i^*(\omega)e^{-j\omega\tau}
\]

(6.22)
where $S_i(\omega)$ is the Fourier transform of $s_i(t)$. Thus, the moduli of $H(\omega)$ and $S_i(\omega)$ are identical; however, the phase responses are opposite of each other.

Example 6.1: Compute the maximum instantaneous SNR at the output of a linear filter whose impulse response is matched to the signal $x(t) = \exp(-t^2/2T)$.

Solution: The signal energy is
\[
E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} e^{-t^2/T} \, dt = \sqrt{\pi T} \text{ Joules}
\]
It follows that the maximum instantaneous SNR is
\[
\text{SNR} = \frac{\sqrt{\pi T}}{N_0/2}
\]
where $N_0/2$ is the input noise power spectrum density.

6.2. The Replica

Again, consider a radar system that uses a finite duration energy signal $s_i(t)$, and assume that a matched filter receiver is utilized. The input signal is given in Eq. (6.1) and is repeated here as Eq. (6.23),
\[
x(t) = C \cdot s_i(t - t_i) + n_i(t)
\]
The matched filter output $y(t)$ can be expressed by the convolution integral between the filter’s impulse response and $x(t)$,
\[
y(t) = \int_{-\infty}^{\infty} x(u)h(t-u) \, du
\]
Substituting Eq. (6.20) into Eq. (6.24) yields
\[
y(t) = \int_{-\infty}^{\infty} x(u)s_i^*(\tau - t + u) \, du = \tilde{R}_{sx}(t - \tau)
\]
where $\tilde{R}_{sx}(t - \tau)$ is a cross-correlation between $x(t)$ and $s_i(\tau - t)$. Therefore, the matched filter output can be computed from the cross-correlation between the radar received signal and a delayed replica of the transmitted waveform. If the input signal is the same as the transmitted signal, the output of the matched

© 2000 by Chapman & Hall/CRC
filter would be the autocorrelation function of the received (or transmitted) signal. In practice, replicas of the transmitted waveforms are normally computed and stored in memory for use by the radar signal processor when needed.

### 6.3. Matched Filter Response to LFM Waveforms

In order to develop a general expression for the matched filter output when an LFM waveform is utilized, we will consider the case when the radar is tracking a closing target with velocity \( v \). The transmitted signal is

\[
s_\gamma(t) = \text{Rect}(\frac{t}{\tau}) e^{j2\pi f_0(t+\frac{\gamma}{2})} \tag{6.26}
\]

The received signal is then given by

\[
s_r(t) = s_\gamma(t-\Delta(t)) \tag{6.27}
\]

\[
\Delta(t) = t_0 - \frac{2v}{c}(t-t_0) \tag{6.28}
\]

where \( t_0 \) is the time corresponding to the target initial detection range, and \( c \) is the speed of light. Using Eq. (6.28) we can rewrite Eq. (6.27) as

\[
s_r(t) = s_\gamma\left(t-t_0 + \frac{2v}{c}(t-t_0)\right) = s_\gamma(\gamma(t-t_0)) \tag{6.29}
\]

and

\[
\gamma = 1 + \frac{2v}{c} \tag{6.30}
\]

is the scaling coefficient. Substituting Eq. (6.26) into Eq. (6.29) yields

\[
s_r(t) = \text{Rect}\left(\frac{\gamma(t-t_0)}{\tau}\right) e^{j2\pi f_0(\gamma(t-t_0))} e^{j\pi k^2(t-t_0)^2} \tag{6.31}
\]

which is the analytical signal representation for \( s_r(t) \). The complex envelope of the signal \( s_r(t) \) is obtained by multiplying Eq. (6.31) by \( \exp(-j2\pi f_0 t) \). Denote the complex envelope by \( s_\gamma(t) \), then after some manipulation we get

\[
s_\gamma(t) = e^{-j2\pi f_0 t_0} \text{Rect}\left(\frac{\gamma(t-t_0)}{\tau}\right) e^{j2\pi f_0(\gamma-1)(t-t_0)} e^{j\pi k^2(t-t_0)^2} \tag{6.32}
\]

The Doppler shift due to the target motion is

© 2000 by Chapman & Hall/CRC
\[ f_d = \frac{2v}{c} f_0 \]  

(6.33)

and since \( \gamma - 1 = \frac{2v}{c} \), we get

\[ f_d = (\gamma - 1)f_0 \]  

(6.34)

Using the approximation \( \gamma = 1 \) and Eq. (6.34), Eq. (6.32) is rewritten as

\[ s_\gamma(t) = e^{j2\pi f_d(t-t_0)} s(t-t_0) \]  

(6.35)

where

\[ s(t-t_0) = e^{-j2\pi f_d} s_1(t-t_0) \]  

(6.36)

\( s_1(t) \) is given in Eq. (6.26). The matched filter response is given by the convolution integral

\[ s_o(t) = \int h(u)s_\gamma(t-u)du \]  

(6.37)

For a non-causal matched filter the impulse response \( h(u) \) is equal to \( s^*(u-t) \); it follows that

\[ s_o(t) = \int s^*(-u)s_\gamma(t-u)du \]  

(6.38)

Substituting Eq. (6.36) into Eq. (6.38), and performing some algebraic manipulations, we get

\[ s_o(t) = \int s^*(u) e^{j2\pi f_d(t+u-t_0)} s(t+u-t_0)du \]  

(6.39)

Finally, making the change of variable \( t' = t + u \) yields

\[ s_o(t) = \int s^*(t' - t)s(t' - t_0)e^{j2\pi f_d(t' - t_0)} dt' \]  

(6.40)

It is customary to set \( t_0 = 0 \), and it follows that

\[ s_o(tfd) = \int s(t')s^*(t' - t)e^{j2\pi f_d} dt' \]  

(6.41)
where we used the notation \( s_\tau(t;f_d) \) to indicate that the output is a function of both time and Doppler frequency.

The two-dimensional (2-D) correlation function for the signal \( s(t) \) is obtained from the matched filter response by replacing \( t \) by \( -\tau \), then

\[
\chi(\tau,f_d) = \int_{-\infty}^{\infty} s(t')s^*(t' + \tau)e^{j2\pi f_d t'} dt'
\]  

(6.42)

6.4. The Radar Ambiguity Function

The radar ambiguity function represents the output of the matched filter, and it describes the interference caused by range and/or Doppler of a target when compared to a reference target of equal RCS. The ambiguity function evaluated at \((\tau, f_d) = (0, 0)\) is equal to the matched filter output that is matched perfectly to the signal reflected from the target of interest. In other words, returns from the nominal target are located at the origin of the ambiguity function. Thus, the ambiguity function at nonzero \( \tau \) and \( f_d \) represents returns from some range and Doppler different from those for the nominal target.

The radar ambiguity function is normally used by radar designers as a means of studying different waveforms. It can provide insight about how different radar waveforms may be suitable for the various radar applications. It is also used to determine the range and Doppler resolutions for a specific radar waveform. The three-dimensional (3-D) plot of the ambiguity function versus frequency and time delay is called the radar ambiguity diagram. The radar ambiguity function for the signal \( s(t) \) is defined as the modulus squared of its 2-D correlation function, i.e., \(|\chi(\tau,f_d)|^2\). More precisely,

\[
|\chi(\tau,f_d)|^2 = \left| \int_{-\infty}^{\infty} s(t)s^*(t + \tau)e^{j2\pi f_d t} dt \right|^2
\]  

(6.43)

In this notation, the target of interest is located at \((\tau, f_d) = (0, 0)\), and the ambiguity diagram is centered at the same point. Note that some authors define the ambiguity function as \(|\chi(\tau,f_d)|\). In this book, \(|\chi(\tau,f_d)|\) is called the uncertainty function. Denote \( E \) as the energy of the signal \( s(t) \),

\[
E = \int_{-\infty}^{\infty} |s(t)|^2 dt
\]  

(6.44)

We will now list the properties for the radar ambiguity function:
1) The maximum value for the ambiguity function occurs at \((\tau, f_d) = (0, 0)\) and is equal to \(4E^2\),

\[
\max \{ |\chi(\tau, f_d)|^2 \} = |\chi(0; 0)|^2 = (2E)^2
\]  
\[
|\chi(\tau; f_d)|^2 \leq |\chi(0; 0)|^2
\]  

2) The ambiguity function is symmetric,

\[
|\chi(\tau; f_d)|^2 = |\chi(-\tau; -f_d)|^2
\]  

3) The total volume under the ambiguity function is constant,

\[
\int \int |\chi(\tau; f_d)|^2 \, d\tau \, df_d = (2E)^2
\]  

4) If the function \(S(f)\) is the Fourier transform of the signal \(s(t)\), then by using Parseval’s theorem we get

\[
|\chi(\tau; f_d)|^2 = \left| \int S^*(f)S(f - f_d)e^{-j2\pi ft} \, df \right|^2
\]  

### 6.5. Examples of the Ambiguity Function

The ideal radar ambiguity function is represented by a spike of infinitesimal width that peaks at the origin and is zero everywhere else, as illustrated in Fig. 6.1. An ideal ambiguity function provides perfect resolution between neighboring targets regardless of how close they may be with respect to each other. Unfortunately, an ideal ambiguity function cannot physically exist. This is because the ambiguity function must have finite peak value equal to \((2E)^2\) and a finite volume also equal to \((2E)^2\). Clearly, the ideal ambiguity function cannot meet those two requirements.

#### 6.5.1. Single Pulse Ambiguity Function

Consider the normalized rectangular pulse \(s(t)\) defined by

\[
s(t) = \frac{1}{\sqrt{\tau}} \text{rect}\left(\frac{t}{\tau}\right)
\]  

From Eq. (6.42) we have

\[
\chi(\tau; f_d) = \int_s(t)s^*_s(t + \tau)e^{j2\pi ft_d} \, dt
\]  

© 2000 by Chapman & Hall/CRC
Substituting Eq. (6.50) into Eq. (6.51) and performing the integration yield,

\[ |\chi(\tau, f_d)|^2 = \left( 1 - \frac{|\tau|}{\tau'} \right)^2 \left| \frac{\sin(\pi f_d (\tau' - |\tau|))}{\pi f_d (\tau' - |\tau|)} \right|^2 \quad |\tau| \leq \tau' \quad (6.52) \]

**MATLAB Function “single_pulse_ambg.m”**

The function “single_pulse_ambg.m” implements Eq. (6.52). It is given in Listing 6.1 in Section 6.7. The syntax is as follows:

```
single_pulse_ambg [taup]
```

`taup` is the pulse width. Fig. 6.2 (a-d) shows 3-D and contour plots of single pulse uncertainty and ambiguity functions. These plots can be reproduced using MATLAB program “fig6_2.m” given in Listing 6.2 in Section 6.7.

The ambiguity function cut along the time delay axis \( \tau \) is obtained by setting \( f_d = 0 \). More precisely,

\[ |\chi(\tau, 0)|^2 = \left( 1 - \frac{|\tau|}{\tau'} \right)^2 \quad |\tau| \leq \tau' \quad (6.53) \]

Note that the time autocorrelation function of the signal \( s(t) \) is equal to \( \chi(\tau, 0) \). Similarly, the cut along the Doppler axis is

\[ |\chi(0, f_d)|^2 = \left| \frac{\sin \pi \tau f_d}{\pi \tau' f_d} \right|^2 \quad (6.54) \]

Figs. 6.3 and 6.4, respectively, show the plots of the uncertainty function cuts defined by Eqs. (6.53) and (6.54). Since the zero Doppler cut along the time delay axis extends between \(-\tau'\) and \(\tau'\), then, close targets would be unambiguous if they are at least \(\tau'\) seconds apart.
Figure 6.2a. Single pulse 3-D uncertainty plot. Pulse width is 2 seconds.

Figure 6.2b. Contour plot corresponding to Fig. 6.2a.
Figure 6.2c. Single pulse 3-D ambiguity plot. Pulse width is 2 seconds.

Figure 6.2d. Contour plot corresponding to Fig. 6.2c.
The zero time cut along the Doppler frequency axis has a \((\sin{x}/x)^2\) shape. It extends from \(-\infty\) to \(+\infty\). The first null occurs at \(f_d = \pm 1/\tau'\). Hence, it is possible to detect two targets that are shifted by \(1/\tau'\), without any ambiguity.

We conclude that a single pulse range and Doppler resolutions are limited by the pulse width \(\tau'\). Fine range resolution requires that a very short pulse be used. Unfortunately, using very short pulses requires very large operating bandwidths, and may limit the radar average transmitted power to impractical values.

\[
\frac{x}{\sin{x}} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{t + \frac{1}{\tau'}} dt
\]

Figure 6.3. Zero Doppler uncertainty function cut along the time delay axis.

Figure 6.4. Uncertainty function of a single frequency pulse (zero delay). This plot can be reproduced using MATLAB program “Fig6_4.m” given in Listing 6.3 in Section 6.7.
6.5.2. LFM Ambiguity Function

Consider the LFM complex envelope signal defined by

\[ s(t) = \frac{1}{\sqrt{\tau^2}} \text{Rect} \left( \frac{t}{\tau} \right) e^{j \pi \mu t^2} \quad (6.55) \]

In order to compute the ambiguity function for the LFM complex envelope, we will first consider the case when \( 0 \leq \tau \leq \tau' \). In this case the integration limits are from \( -\tau' / 2 \) to \( (\tau' / 2) - \tau \). Substituting Eq. (6.55) into Eq. (6.51) yields

\[ \chi(\tau; f_d) = \frac{1}{\tau'} \int \text{Rect} \left( \frac{t}{\tau} \right) \text{Rect} \left( \frac{t + \tau}{\tau'} \right) e^{j \pi \mu t^2} e^{-j \pi \mu (t + \tau)^2} \, dt \quad (6.56) \]

It follows that

\[ \chi(\tau; f_d) = \frac{e^{-j \pi \mu \tau^2}}{\tau'} \int e^{-j 2 \pi (\tau - f_d) t} \, dt \quad (6.57) \]

We will leave the rest of the integration process to the reader. Finishing the integration process in Eq. (6.57) yields

\[ \chi(\tau; f_d) = e^{j \pi \mu \tau} \left( 1 - \frac{\tau}{\tau'} \right) \frac{\sin \left( \pi \tau (\mu \tau + f_d) \left( 1 - \frac{\tau}{\tau'} \right) \right)}{\pi \tau (\mu \tau + f_d) \left( 1 - \frac{\tau}{\tau'} \right)} \quad 0 \leq \tau \leq \tau' \quad (6.58) \]

Similar analysis for the case when \( -\tau' \leq \tau \leq 0 \) can be carried out, where in this case the integration limits are from \( (-\tau' / 2) - \tau \) to \( \tau' / 2 \). The same result can be obtained by using the symmetry property of the ambiguity function \( \chi(-\tau, -f_d) = \chi(\tau, f_d) \). It follows that an expression for \( \chi(\tau; f_d) \) that is valid for any \( \tau \) is given by

\[ \chi(\tau; f_d) = e^{j \pi \mu t} \left( 1 - \frac{|t|}{\tau'} \right) \frac{\sin \left( \pi \tau (\mu \tau + f_d) \left( 1 - \frac{|t|}{\tau'} \right) \right)}{\pi \tau (\mu \tau + f_d) \left( 1 - \frac{|t|}{\tau'} \right)} \quad |t| \leq \tau' \quad (6.59) \]

and the LFM ambiguity function is
\[ |\chi(\tau; f_d)|^2 = \left(1 - \frac{|\tau|}{\tau'}\right) \frac{\sin\left(\pi\tau (\mu \tau + f_d)\left(1 - \frac{|\tau|}{\tau'}\right)\right)}{\pi\tau (\mu \tau + f_d)\left(1 - \frac{|\tau|}{\tau'}\right)} |\tau| \leq \tau' \quad (6.60) \]

Again the time autocorrelation function is equal to \( \chi(\tau, 0) \). The reader can verify that the ambiguity function for a down-chirp LFM waveform is given by

\[ |\chi(\tau; f_d)|^2 = \left(1 - \frac{|\tau|}{\tau'}\right) \frac{\sin\left(\pi\tau (\mu \tau - f_d)\left(1 - \frac{|\tau|}{\tau'}\right)\right)}{\pi\tau (\mu \tau - f_d)\left(1 - \frac{|\tau|}{\tau'}\right)} |\tau| \leq \tau' \quad (6.61) \]

**MATLAB Function “lfm Ambg.m”**

The function “lfm Ambg.m” implements Eqs. (6.60) and (6.61). It is given in Listing 6.4 in Section 6.7. The syntax is as follows:

\[ \text{lfm Ambg [taup, b, up_down]} \]

where

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Units</th>
<th>Status</th>
</tr>
</thead>
<tbody>
<tr>
<td>taup</td>
<td>pulse width</td>
<td>seconds</td>
<td>input</td>
</tr>
<tr>
<td>b</td>
<td>bandwidth</td>
<td>Hz</td>
<td>input</td>
</tr>
<tr>
<td>up_down</td>
<td>up_down = 1 for up chirp</td>
<td>none</td>
<td>input</td>
</tr>
<tr>
<td></td>
<td>up_down = -1 for down chirp</td>
<td>none</td>
<td>input</td>
</tr>
</tbody>
</table>

Fig. 6.5 (a-d) shows 3-D and contour plots for the LFM uncertainty and ambiguity functions for

<table>
<thead>
<tr>
<th>taup</th>
<th>b</th>
<th>up_down</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>1</td>
</tr>
</tbody>
</table>

These plots can be reproduced using MATLAB program “fig6_5.m” given in Listing 6.5 in Section 6.7. This function generates 3-D and contour plots of an LFM ambiguity function.

The up-chirp ambiguity function cut along the time delay axis \( \tau \) is

\[ |\chi(\tau; 0)|^2 = \left(1 - \frac{|\tau|}{\tau'}\right) \frac{\sin\left(\pi\mu \tau^2\left(1 - \frac{|\tau|}{\tau'}\right)\right)}{\pi\mu \tau^2\left(1 - \frac{|\tau|}{\tau'}\right)} |\tau| \leq \tau' \quad (6.62) \]
Figure 6.5a. Up-chirp LFM 3-D uncertainty plot. Pulse width is 1 second; and bandwidth is 10 Hz.

Figure 6.5b. Contour plot corresponding to Fig. 6.5a.
Figure 6.5c. Up-chirp LFM 3-D ambiguity plot. Pulse width is 1 second; and bandwidth is 10 Hz.

Figure 6.5d. Contour plot corresponding to Fig. 6.5c.
Fig. 6.6 shows a plot for a cut in the uncertainty function corresponding to Eq. (6.62). Note that the LFM ambiguity function cut along the Doppler frequency axis is similar to that of the single pulse. This should not be surprising since the pulse shape has not changed (we only added frequency modulation). However, the cut along the time delay axis changes significantly. It is now much narrower compared to the unmodulated pulse cut. In this case, the first null occurs at

\[ \tau_{n1} = 1/B \]  

which indicates that the effective pulse width (compressed pulse width) of the matched filter output is completely determined by the radar bandwidth. It follows that the LFM ambiguity function cut along the time delay axis is narrower than that of the unmodulated pulse by a factor

\[ \xi = \frac{\tau'}{(1/B)} = \tau'B \]  

\( \xi \) is referred to as the compression ratio (also called time-bandwidth product and compression gain). All three names can be used interchangeably to mean the same. As indicated by Eq. (6.64) the compression ratio also increases as the radar bandwidth is increased.

Figure 6.6. Zero Doppler Ambiguity function of an LFM pulse (\( \tau' = 1, \ b = 20 \)). This plot can be reproduced using MATLAB program “fig6_6.m” given in Listing 6.6 in Section 6.7.
Example 6.2: Compute the range resolution before and after pulse compression corresponding to an LFM waveform with the following specifications: Bandwidth \( B = 1 \text{GHz} \); and pulse width \( \tau' = 10 \text{ms} \).

Solution: The range resolution before pulse compression is

\[
\Delta R_{\text{uncomp}} = \frac{c \tau'}{2} = \frac{10 \times 10^{-3} \times 3 \times 10^{8}}{2} = 1.5 \times 10^{6} \text{ meters}
\]

Using Eq. (6.63) yields

\[
\tau_{n1} = \frac{1}{1 \times 10^{9}} = 1 \text{ ns}
\]

\[
\Delta R_{\text{comp}} = \frac{c \tau_{n1}}{2} = \frac{3 \times 10^{8} \times 1 \times 10^{-9}}{2} = 15 \text{ cm}.
\]

6.5.3. Coherent Pulse Train Ambiguity Function

Fig. 6.7 shows a plot of coherent pulse train. The pulse width is denoted as \( \tau' \) and the PRI is \( T \). The number of pulses in the train is \( N \); hence, the train’s length is \( (N - 1)T \) seconds. A normalized individual pulse \( s(t) \) is defined by

\[
s_{1}(t) = \frac{1}{\sqrt{\tau'}} \text{Rect}\left(\frac{t}{\tau'}\right)
\]  

(6.65)

When coherency is maintained between the consecutive pulses, then an expression for the normalized train is

\[
s(t) = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} s_{1}(t - iT)
\]  

(6.66)

The output of the matched filter is

\[
\chi(\tau; f_{d}) = \int_{-\infty}^{\infty} s(t)s^{*}(t + \tau)e^{j2\pi f_{d}t} \, dt
\]  

(6.67)

Substituting Eq. (6.66) into Eq. (6.67) and interchanging the summations and integration yield,

\[
\chi(\tau; f_{d}) = \frac{1}{N} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \int_{-\infty}^{\infty} s_{1}(t - iT) s_{1}^{*}(t - jT - \tau)e^{j2\pi f_{d}t} \, dt
\]  

(6.68)
Making the change of variable \( t_1 = t - iT \) yields

\[
\chi(\tau;f_d) = \frac{1}{N} \sum_{i=0}^{N-1} e^{j2\pi f_d iT} \sum_{j=0}^{N-1} \int s_1(t_i) s_1^*(t - \tau_j) e^{j2\pi f_d t_i} dt_i \tag{6.69}
\]

The integral inside Eq. (6.69) represents the output of the matched filter for a single pulse, and is denoted by \( \chi_1 \). It follows that

\[
\chi(\tau;f_d) = \frac{1}{N} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \chi_1[\tau_j - (i - j)T] \tag{6.70}
\]

When the relation \( q = i - j \) is used, then the following relation is true:

\[
\sum_{j=0}^{N-1} \sum_{m=0}^{N-1} = \sum_{q=-(N-1)}^{N-1} \sum_{i=0}^{N-1} = \sum_{j=0}^{N-1} \sum_{q=1}^{N-1} \text{for } j = i - q \quad \sum_{j=0}^{N-1} \sum_{q=0}^{N-1} \text{for } i = j + q
\]

Using Eq. (6.71) into Eq. (6.70) gives

\[
\chi(\tau;f_d) = \frac{1}{N} \sum_{q=-(N-1)}^{N-1} \chi_1(\tau - qT;f_d) \sum_{i=0}^{N-1} e^{j2\pi f_d iT} \tag{6.72}
\]


© 2000 by Chapman & Hall/CRC
Setting $z = \exp(j2\pi f_d T)$, and using the relation

$$
\sum_{j=0}^{N-1-|q|} z^j = \frac{1-z^{N-|q|}}{1-z}
$$

yield

$$
\sum_{i=0}^{N-1-|q|} e^{j2\pi f_d i T} = e^{j|\pi f_d (N-1-|q|)T|} \frac{\sin[\pi f_d (N-1-|q|)T]}{\sin(\pi f_d T)}
$$

Using Eq. (6.74) into Eq. (6.72) yields two complementary sums for positive and negative $q$. Both sums can be combined as

$$
\chi(\tau f_d) = \frac{1}{N} \sum_{q=-(N-1)}^{N-1} \chi_i(\tau - q T; f_d) e^{j|\pi f_d (N-1+q)T|} \frac{\sin[\pi f_d (N-|q|)T]}{\sin(\pi f_d T)}
$$

Finally, the ambiguity function associated with the coherent pulse train is computed as the modulus square of Eq. (6.75). For $\tau' < T/2$, the ambiguity function reduces to

$$
\chi(\tau f_d) = \frac{1}{N} \sum_{q=-(N-1)}^{N-1} |\chi_i(\tau - q T; f_d)| \frac{\sin[\pi f_d (N-|q|)T]}{\sin(\pi f_d T)}
$$

Thus, the ambiguity function for a coherent pulse train is the superposition of the individual pulse’s ambiguity functions. The ambiguity function cuts along the time delay and Doppler axes are, respectively, given by

$$
|\chi(\tau;0)|^2 = \sum_{q=-(N-1)}^{N-1} \left(1 - \frac{|q|}{N}\right) \left(1 - \frac{\tau - q T}{\tau'}\right) : |\tau - q T| < \tau'
$$

$$
|\chi(0;\tau')|^2 = \left(\frac{1}{N} \frac{\sin(\pi f_d \tau')}{\pi f_d \tau'} \frac{\sin(\pi f_d NT)}{\sin(\pi f_d T)}\right)^2
$$

**MATLAB Function “train_ambg.m”**

The function “train_ambg.m” implements Eq. (6.76). It is given in Listing 6.7 in Section 6.7. The syntax is as follows:

```
train_ambg [taup, n, pri]
```
where

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Units</th>
<th>Status</th>
</tr>
</thead>
<tbody>
<tr>
<td>taup</td>
<td>pulse width</td>
<td>seconds</td>
<td>input</td>
</tr>
<tr>
<td>n</td>
<td>number of pulses in train</td>
<td>none</td>
<td>input</td>
</tr>
<tr>
<td>pri</td>
<td>pulse repetition interval</td>
<td>seconds</td>
<td>input</td>
</tr>
</tbody>
</table>

Fig. 6.8 (a-d) shows typical outputs of this function, for

<table>
<thead>
<tr>
<th>taup</th>
<th>n</th>
<th>pri</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

Figure 6.8a. Three-dimensional ambiguity plot for a five pulse equal amplitude coherent train. Pulse width is 0.2 seconds; and PRI is 1 second, N=5. This plot can be reproduced using MATLAB program fig6_8a.m given in Listing 6.8 in Section 6.7.
Figure 6.8b. Contour plot corresponding to Fig. 6.8a.

Figure 6.8c. Zero Doppler cut corresponding to Fig. 6.8a.
6.6. Ambiguity Diagram Contours

Plots of the ambiguity function are called ambiguity diagrams. For a given waveform, the corresponding ambiguity diagram is normally used to determine the waveform properties such as the target resolution capability, measurements (time and frequency) accuracy and its response to clutter. Three-dimensional ambiguity diagrams are difficult to plot and interpret. This is the reason why contour plots of the 3-D ambiguity diagram are often used to study the characteristics of a waveform. An ambiguity contour is a 2-D plot (frequency/time) of a plane intersecting the 3-D ambiguity diagram that corresponds to some threshold value. The resultant plots are ellipses. It is customary to display the ambiguity contour plots that correspond to one half of the peak autocorrelation value.

Fig. 6.9 shows a sketch of typical ambiguity contour plots associated with a gated CW pulse. It indicates that narrow pulses provide better range accuracy than long pulses. Alternatively, the Doppler accuracy is better for a wider pulse than it is for a short one. This trade-off between range and Doppler measurements comes from the uncertainty associated with the time-bandwidth product of a single sinusoidal pulse, where the product of uncertainty in time (range) and uncertainty in frequency (Doppler) cannot be much smaller than unity. Note that an exact plot for Fig. 6.9 can be obtained using the function “single_pulse_ambg.m” and the MATLAB command contour.

© 2000 by Chapman & Hall/CRC
Multiple ellipses in an ambiguity contour plot indicate the presence of multiple targets. Thus, it seems that one may improve the radar resolution by increasing the ambiguity diagram threshold value. This is illustrated in Fig. 6.10. However, in practice this is not possible for two reasons. First, in the presence of noise we lack knowledge of the peak correlation value; and second, targets in general will have different amplitudes.

Now consider the case of a coherent pulse train described in Fig. 6.7. For a pulse train, range accuracy is still determined by the pulse width, the same way as in the case of a single pulse, while Doppler accuracy is determined by the train length. Thus, time and frequency measurements can be made independently of each other. However, additional peaks appear in the ambiguity diagram which may cause range and Doppler uncertainties. This is illustrated in Fig. 6.11.

Figure 6.9. Ambiguity contour plot associated with a sinusoid modulated gated CW pulse. See Fig. 6.2.

Figure 6.10. Effect of threshold value on resolution.
As one would expect, high PRF pulse trains (i.e., small $T$) lead to extreme uncertainty in range, while low PRF pulse trains have extreme ambiguity in Doppler, as shown in Fig. 6.12. Medium PRF pulse trains have moderate ambiguity in both range and Doppler, which can be overcome by using multiple PRFs, as illustrated in Fig. 6.13 for two medium PRFs. Note that the two diagrams (in Fig. 6.13) agree only in one location (center of the plot) which corresponds to the true target location.

It is possible to avoid ambiguities caused by pulse trains and still have reasonable independent control on both range and Doppler accuracies by using a single modulated pulse with a time-bandwidth product that is much larger than unity. Figure 6.14 shows the ambiguity contour plot associated with an LFM waveform. In this case, $\tau'$ is the pulse width and $B$ is the pulse bandwidth. In this case, exact plots can be obtained using the function “lfm_ambg.m”.

Figure 6.11. Ambiguity contour plot corresponding to Fig. 6.7. For an exact plot see Fig. 6.8b.

© 2000 by Chapman & Hall/CRC
**Figure 6.12.** Uncertainty associated with low and high PRFs.

**Figure 6.13.** Uncertainty of two medium PRFs.
6.7. MATLAB Listings

This section presents listings for all MATLAB programs/functions used in this chapter. The user is strongly advised to rerun the MATLAB programs in order to enhance their understanding of this chapter’s material.

---

**Listing 6.1. MATLAB Function “single_pulse_ambg.m”**

```matlab
function x = single_pulse_ambg (taup)
colormap (gray(1))
eps = 0.000001;
i = 0;
taumax = 1.1 * taup;
taumin = -taumax;
for tau = taumin:.05:taumax
    i = i + 1;
    j = 0;
    for fd = -5/taup:.05:5/taup
        j = j + 1;
        val1 = 1 - abs(tau) / taup;
        val2 = pi * taup * (1.0 - abs(tau) / taup) * fd;
        x(j,i) = abs( val1 * sin(val2+eps)/(val2+eps));
    end
end
```

---

**Listing 6.2. MATLAB Program “fig6_2.m”**

```matlab
clear all
eps = 0.000001;
```

---

Figure 6.14. Ambiguity contour plot associated with an up-chirp LFM waveform. For an exact plot see Fig. 6.5b.
taup = 2.;
taumin = -1.1 * taup;
taumax = -taumin;
x = single_pulse_ambg(taup);
taux = taumin:.05:taumax;
fdy = -5/taup:.05:5/taup;
figure(1)
mesh(taux,fdy,x);
xlabel ('Delay - seconds')
ylabel ('Doppler - Hz')
zlabel ('Ambiguity function')
figure(2)
contour(taux,fdy,x);
xlabel ('Delay - seconds')
ylabel ('Doppler - Hz')
y = x.^2;
figure(3)
mesh(taux,fdy,y);
xlabel ('Delay - seconds')
ylabel ('Doppler - Hz')
zlabel ('Ambiguity function')
figure(4)
contour(taux,fdy,y);
xlabel ('Delay - seconds')
ylabel ('Doppler - Hz')

clear all
eps = 0.0001;
taup = 2.;
fd = -10./taup:.05:10./taup;
uncer = abs(sinc(taup .* fd));
ambg = uncer.^2;
plot(fd, ambg)
xlabel ('Frequency - Hz')
ylabel ('Ambiguity - Volts')
grid
figure(2)
plot (fd, uncer);
xlabel ('Frequency - Hz')
ylabel ('Uncertainty - Volts')
grid

function x = lfm_ambg(taup, b, up_down)
eps = 0.000001;
Listing 6.5. MATLAB Program “fig6_5.m”

```matlab
clear all
eps = 0.0001;
taup = 1.;
b =10.;
up_down = 1.;
x = lfm_ambg(taup, b, up_down);
taux = -1.1*taup:.05:1.1*taup;
fdy = -b:.05:b;
figure(1)
mesh(taux,fdy,x)
xlabel ('Delay - seconds')
ylabel ('Doppler - Hz')
zlabel ('Ambiguity function')
figure(2)
contour(taux,fdy,x)
xlabel ('Delay - seconds')
ylabel ('Doppler - Hz')
y = sqrt(x);
figure(3)
mesh(taux,fdy,y)
xlabel ('Delay - seconds')
ylabel ('Doppler - Hz')
zlabel ('Uncertainty function')
figure(4)
contour(taux,fdy,y)
xlabel ('Delay - seconds')
ylabel ('Doppler - Hz')
```

Listing 6.6. MATLAB Program “fig6_6.m”

```matlab
clear all
```

© 2000 by Chapman & Hall/CRC
taup = 1;
b = 20.;
up_down = 1.;
taux = -1.5*taup:.01:1.5*taup;
fd = 0.;
mu = up_down * b / 2. / taup;
ii = 0.;
for tau = -1.5*taup:.01:1.5*taup
  ii = ii + 1;
  val1 = 1. - abs(tau) / taup;
  val2 = pi * taup * (1.0 - abs(tau) / taup);
  val3 = (fd + mu * tau);
  val = val2 * val3;
  x(ii) = abs( val1 * (sin(val+eps)/(val+eps)));
end
figure(1)
plot(taux,x)
grid
xlabel ('Delay - seconds')
ylabel ('Uncertainty')
figure(2)
plot(taux,x.^2)
grid
xlabel ('Delay - seconds')
ylabel ('Ambiguity')

Listing 6.7. MATLAB Function “train_ambg.m”

function x = train_ambg (taup, n, pri)
if (taup > pri / 2.)
  'ERROR. Pulse width must be less than the PRI/2.'
  break
end
gap = pri - 2.*taup;
eps = 0.000001;
b = 1. / taup;
ii = 0.;
for q = -(n-1):1:n-1
  tao = q - taup;
  index = -1.;
  for tau = tao:0.0533:tao+gap+2.*taup
    index = index + 1;
    tau = -taup + index*.0533;
    ii = ii + 1;
    j = 0.;
    for fd = -b:.0533:b
      j = j + 1;
      if (abs(tau) <= taup)
val1 = 1.0 -abs(tau) / taup;
val2 = pi * taup * fd * (1.0 - abs(tau) / taup);
val3 = abs(val1 * sin(val2+eps) / (val2+eps));
val4 = abs((sin(pi*fd*(n-abs(q))*pri+eps)) / (sin(pi*fd*pri+eps)));
x(j,ii)= val3 * val4 / n;
else
x(j,ii) = 0.;
end
end
end

Listing 6.8. MATLAB Program “fig6_8a.m”

clear all
taup = 0.2;
pri=1;
n=5;
x = train_ambg (taup, n, pri);
figure(1)
mesh(x)
xlabel ('Delay - seconds')
ylabel ('Doppler - Hz')
zlabel ('Ambiguity function')
figure(2)
contour(x);
xlabel ('Delay - seconds')
ylabel ('Doppler - Hz')

Problems

6.1. Define \{x_I(n) = 1, -1, 1\} and \{x_Q(n) = 1, 1, -1\}. (a) Compute the discrete correlations: \(R_{x_I} \), \(R_{x_Q} \), \(R_{x_IX_Q}\), and \(R_{x_Qx_I} \). (b) A certain radar transmits the signal \(s(t) = x_I(t)\cos2\pi f_0 t - x_Q(t)\sin2\pi f_0 t \). Assume that the autocorrelation \(s(t)\) is equal to \(y(t) = y_I(t)\cos2\pi f_0 t - y_Q(t)\sin2\pi f_0 t \). Compute and sketch \(y_I(t)\) and \(y_Q(t)\).

6.2. Compute the frequency response for the filter matched to the signal

\[ (a) \ x(t) = \exp\left(-\frac{t^2}{2\tau^2}\right); \ (b) \ x(t) = u(t)\exp(-\alpha t), \]

where \(\alpha\) is a positive constant.

6.3. Repeat Example 6.1 for \(x(t) = u(t)\exp(-\alpha t).\)
6.4. Derive Eq. (6.43).
6.5. Prove the properties of the radar ambiguity function.
6.7. A radar system uses LFM waveforms. The received signal is of the form \( s(t) = As(t - \tau) + n(t) \), where \( \tau \) is a time delay that depends on range, \( s(t) = \text{Rect}(t/\tau')\cos(2\pi f_0 t - \psi(t)) \), and \( \psi(t) = -\pi B t^2/\tau' \). Assume that the radar bandwidth is \( B = 5 \text{MHz} \), and the pulse width is \( \tau' = 5 \mu s \). (a) Give the quadrature components of the matched filter response that matched to \( s(t) \). (b) Write an expression for the output of the matched filter. (c) Compute the increase in SNR produced by the matched filter.
6.8. (a) Write an expression for the ambiguity function of an LFM waveform, where \( \tau' = 6.4 \mu s \), and the compression ratio is 32. (b) Give an expression for the matched filter impulse response.
6.9. Repeat Example 6.2 for \( B = 2, 5 \), and 10 \text{GHz}.
6.10. (a) Write an expression for the ambiguity function of a LFM signal with bandwidth \( B = 10 \text{MHz} \), pulse width \( \tau' = 1 \mu s \), and wavelength \( \lambda = 1 \text{cm} \). (b) Plot the zero Doppler cut of the ambiguity function. (c) Assume a target moving towards the radar with radial velocity \( v_r = 100 \text{m/s} \). What is the Doppler shift associated with this target? (d) Plot the ambiguity function for the Doppler cut in part (c). (e) Assume that three pulses are transmitted with PRF \( f_r = 2000 \text{Hz} \). Repeat part b.
6.11. (a) Give an expression for the ambiguity function for a pulse train consisting of 4 pulses, where the pulse width is \( \tau' = 1 \mu s \) and the pulse repetition interval is \( T = 10 \mu s \). Assume a wavelength of \( \lambda = 1 \text{cm} \). (b) Sketch the ambiguity function contour.
6.12. Hyperbolic frequency modulation (HFM) is better than LFM for high radial velocities. The HFM phase is
\[
\psi_h(t) = \frac{\omega_0^2}{\mu_h} \ln \left( 1 + \frac{\mu_h \alpha f}{\omega_0} \right)
\]
where \( \mu_h \) is an HFM coefficient and \( \alpha \) is a constant. (a) Give an expression for the instantaneous frequency of a HFM pulse of duration \( \tau'_h \). (b) Show that HFM can be approximated by LFM. Express the LFM coefficient \( \mu_f \) in terms of \( \mu_h \) and in terms of \( B \) and \( \tau' \).
6.13. Consider a Sonar system with range resolution \( \Delta R = 4 \text{cm} \). (a) A sinusoidal pulse at frequency \( f_0 = 100 \text{KHz} \) is transmitted. What is the pulse width, and what is the bandwidth? (b) By using an up-chirp LFM, centered at
A pulse train $y(t)$ is given by

$$y(t) = \sum_{n=0}^{2} w(n)x(t-n\tau')$$

where $x(t) = \exp(-t^2/2)$ is a single pulse of duration $\tau'$ and the weighting sequence is $\{w(n)\} = \{0.5, 1, 0.7\}$. Find and sketch the correlations $R_x$, $R_w$, and $R_y$.

6.15. Repeat the previous problem for $x(t) = \exp(-t^2/2)\cos 2\pi f_0 t$.

6.16. Modify the function “train_ambg.m” to accommodate the case $\tau' = T$.

6.17. Using the MATLAB functions presented in this chapter, generate the exact plots that correspond to Figs. 6.13 and 6.14.

6.18. Using the function “lfm_ambg.m” reproduce Fig. 6.6b for a down-chirp LFM pulse.